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Evolution of Burgers' turbulence in the presence of external forces

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Statistical properties of multidimensional Burgers' turbulence evolving in the presence of a force field with random potential, which is delta-correlated in time and smooth in space, are studied in the inviscid limit and at the physical level of rigorousness. The solution algorithm reduces to finding multistream fields describing the motion of an auxiliary gas of interacting particles in a force field. Consequently, the statistical description of forced Burgers' turbulence is obtained by finding the largest possible value of the least action for the auxiliary gas. The exponential growth of the number of streams is found to be a necessary condition for the existence of stationary regimes.

1. Introduction

The search for statistical laws governing the evolution of strongly nonlinear turbulent fields and, in particular, for equilibrium states of these fields with time-invariant statistical properties, remains an important unsolved problem in statistical hydrodynamics. Thus, it is natural to study the same questions for a widely-used mathematical model of real turbulence known as Burgers' turbulence, i.e. the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \mathbf{R}^d, d \geqslant 1$, satisfying the multidimensional Burgers' equation

$$\left.\begin{aligned} \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} &= \mu \Delta \boldsymbol{v} + \boldsymbol{f}(\boldsymbol{x}, t), \\ \boldsymbol{v}(\boldsymbol{x}, t=0) &= \boldsymbol{v}_0(\boldsymbol{x}), \end{aligned}\right\} \tag{1.1}$$

where $\mu > 0$, and the initial velocity $\boldsymbol{v}_0$ along with force $\boldsymbol{f}$ are known and random. Burgers' turbulence is considered as an adequate model of certain aspects of hydrodynamic turbulence since, as was observed by Burgers himself (see e.g. Burgers 1974), it takes into account the competition of two most important mechanisms for real turbulence: inertial nonlinearity and viscous dissipation. There exist, however, some discrepancies between Burgers' and hydrodynamic turbulence which are aggravated by the fact that hydrodynamic turbulence has, primarily, a rotational character, whereas by Burgers' turbulence we usually mean the potential velocity field

$$\boldsymbol{v}(\boldsymbol{x}, t) = \nabla S(\boldsymbol{x}, t), \tag{1.2}$$

generated by potential S, which then satisfies the Hamilton–Jacobi-type equation

$$\frac{\partial S}{\partial t} + \tfrac{1}{2}(\nabla S)^2 = \mu \Delta S + U(\boldsymbol{x}, t), \tag{1.3}$$

where U is the potential of external forces, i.e.

$$\boldsymbol{f}(\boldsymbol{x},t) = \nabla U(\boldsymbol{x},t). \tag{1.4}$$

Nevertheless, the fondness of theoreticians for potential solutions of Burgers' equation can be justified by the existence of the explicit formulas and thus, a realistic hope of quantitative analysis of a strongly nonlinear phenomenon. As was observed by Kraichnan (1959, 1968), the differences between Burgers' and hydrodynamic turbulence are as instructive as the similarities. Also, by now, Burgers' equation has become one of the common nonlinear model equations of mathematical physics and over a period of time has been discovered that such, or similar, models describe various physical phenomena displaying shock formation. For example, during the last decade astrophysicists studying the large-scale distribution of matter in the Universe have convincingly demonstrated that Burgers' turbulence provides an adequate description of the process of formation of cellular structures (see e.g. Gurbatov, Malakhov & Saichev 1991; Shandarin & Zeldovich 1989; Sahni, Sathyaprakash & Shandarin 1994).

The goal of the present article is to provide a quantitative study of the statistically stationary regimes in Burgers' turbulence. To begin with, let us review some conditions for the existence of such equilibria.

Since dissipation leads to a decay of turbulence, to sustain it one needs a supply of energy from outside. In hydrodynamic turbulence in nature such an 'engine' is often solar energy, which generates large-scale convective eddies. Their nonlinear descending cascade maintains in dynamic equilibrium even smaller-scale, turbulent rotational motions.

In Burgers' turbulence (1.1) the necessary input of energy is provided by the external random force field $\boldsymbol{f}(\boldsymbol{x},t)$. Observe, however, that not all force fields $\boldsymbol{f}(\boldsymbol{x},t)$, even if they are stationary in time and homogeneous in space, will lead to a stationary regime in Burgers' turbulence. For that reason one would like to know conditions on forces $\boldsymbol{f}(\boldsymbol{x},t)$ which would guarantee the establishment of a stationary regime as $t \to \infty$. A significant result in this direction has been obtained by Sinai (1991) (see also Sinai 1996) who gave a rigorous proof of the fact that (in the one-dimensional case) there exists a broad class of random potentials $U(\boldsymbol{x},t)$, periodic in space and delta-correlated in time, for which the solution $\boldsymbol{v}(\boldsymbol{x},t)$ of the Burgers' equation converges (as $t \to \infty$) to a solution $\boldsymbol{v}_\infty(\boldsymbol{x},t)$ which is independent of the initial condition, stationary in time and periodic in space. So much for the *positive* results.

On the other hand, *negative* examples abound and, below, we display a case of random forces $\boldsymbol{f}(\boldsymbol{v},t)$ for which the stationary regime is impossible in principle. We shall restrict ourselves here to the one-dimensional Burgers' equation

$$\left.\begin{array}{c} \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = \mu\dfrac{\partial^2 v}{\partial x^2} + f(x,t), \\ v(x,t=0) = v_0(x), \end{array}\right\} \tag{1.5}$$

for the velocity field $v(x,t)$, where $v_0(x)$ is a statistically homogeneous stochastic process with zero mean and correlation function

$$\Gamma_0(z) = \langle v_0(x)v_0(x+z)\rangle, \tag{1.6}$$

and $f(x,t)$ is a Gaussian, delta-correlated in time and statistically homogeneous in space, random field with correlation function

$$\langle f(x,t)f(x+z,t+\tau)\rangle = \Gamma_f(z)\delta(\tau). \tag{1.7}$$

Above, and in what follows, the angled brackets denote statistical averaging over the ensemble of realizations of the force and (if necessary) of the random initial data, which are assumed to be independent of each other.

The spatial correlation function

$$\Gamma(z;t) = \langle v(x,t)v(x+z,t)\rangle,$$

of the one-dimensional Burgers' turbulence satisfies equation

$$\frac{\partial}{\partial t}\Gamma(z;t) + \frac{1}{2}\frac{\partial}{\partial z}\Big[\Gamma_{12}(z;t) - \Gamma_{12}(-z;t)\Big]$$
$$= 2\mu\frac{\partial^2}{\partial z^2}\Gamma(z;t) + \langle f(x,t)\,v(x+z,t)\rangle + \langle f(x+z,t)\,v(x,t)\rangle, \quad \Gamma(z;t=0) = \Gamma_0(z), \tag{1.8}$$

where the third-order moments

$$\Gamma_{12}(z;t) = \langle v(x,t)\,v^2(x+z,t)\rangle.$$

Utilizing the Furutsu–Novikov–Donsker formula (see, Appendix A and e.g. Klyatskin, Woyczynski & Gurarie 1996*a*, *b*, where a complete proof is provided), for the cross-correlations in (1.8) we have

$$\langle f(x,t)\,v(x+z,t)\rangle = \langle f(x+z,t)\,v(x,t)\rangle = \tfrac{1}{2}\Gamma_f(z).$$

As a result, equation (1.8) assumes the form

$$\frac{\partial}{\partial t}\Gamma(z;t) + \frac{1}{2}\frac{\partial}{\partial z}\Big[\Gamma_{12}(z;t) - \Gamma_{12}(-z;t)\Big] = 2\mu\frac{\partial^2}{\partial z^2}\Gamma(z;t) + \Gamma_f(z). \tag{1.9}$$

Introducing the spatial spectral density

$$G(\kappa;t) = \frac{1}{2\pi}\int \Gamma(z;t)\mathrm{e}^{\mathrm{i}\kappa z}\mathrm{d}z$$

of the Burgers' turbulence v, and the spatial spectral density

$$G_f(\kappa) = \frac{1}{2\pi}\int \Gamma_f(z)\mathrm{e}^{\mathrm{i}\kappa z}\mathrm{d}z,$$

of the force field f, we discover from (1.9) and with the help of an assumption natural from the physical viewpoint

$$\lim_{|z|\to\infty}\Gamma_{12}(z,t) = 0,$$

that, at $\kappa = 0$, the former satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}G(0;t) = G_f(0), \tag{1.10}$$

with the initial condition

$$G(0;t=0) = \frac{1}{2\pi}\int \Gamma_0(z)\mathrm{d}z. \tag{1.11}$$

The solution of equation (1.10) is

$$G(0;t) = G(0,t=0) + G_f(0)t, \tag{1.12}$$

so that if $G_f(0) \neq 0$ then the spectral density of Burgers' turbulence grows linearly in time at $\kappa = 0$, which is clearly impossible in a stationary regime. Thus, we have arrived at the following result:

(i) *A necessary condition for the existence of a stationary regime in forced Burgers' turbulence is that*

$$G_f(\kappa = 0) = \frac{1}{2\pi}\int \Gamma_f(z)\,\mathrm{d}z = 0, \tag{1.13}$$

i.e. that the spectral density of the external force vanishes for $\kappa = 0$.

This condition and its multidimensional analogue are fulfilled, in particular, if the *random force's potential* $U(\boldsymbol{x}, t)$ *is statistically homogeneous in space*, and we will make this assumption in the remainder of this paper.

It also follows from (1.12) that the spectral density of Burgers' turbulence depends on $G(0, t = 0)$. This means that if $G(0, t = 0) \neq 0$ then Burgers' turbulence 'always remembers' the initial field. Consequently:

(ii) *A necessary condition for the stationary regime in forced Burgers' turbulence to be ergodic (i.e. independent of the initial field) is that*

$$G(0; t = 0) = \frac{1}{2\pi}\int \Gamma_0(z)\mathrm{d}z = 0. \tag{1.14}$$

Observe that the necessary conditions (1.13)–(1.14) for the existence of an ergodic stationary regime are clearly satisfied for the class of forces and initial conditions studied by Sinai (1991).

Equation (1.9) also permits us to formulate the following, somewhat less obvious, result about statistical properties of stationary regimes in forced Burgers' turbulence. Its validity follows directly from (1.12)–(1.14).

(iii) *Assume that there exists an ergodic stationary regime of Burgers' turbulence and that the limits*

$$\Gamma^{\infty}(z) = \lim_{t\to\infty} \Gamma(z, t), \qquad \Gamma_{12}^{\infty}(z) = \lim_{t\to\infty} \Gamma_{12}(z, t)$$

exist. Then, its spectral density vanishes at $\kappa = 0$, *i.e.*

$$G^{\infty}(\kappa = 0) = \frac{1}{2\pi}\int \Gamma^{\infty}(z)\,\mathrm{d}z = 0. \tag{1.15}$$

Other propositions will answer the question of whether a Gaussian stationary regime is feasible. To arrive at these results, observe that in the stationary regime, equation (1.9) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}z}\Gamma_{12:odd}^{\infty}(z) = 2\mu\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Gamma^{\infty}(z) + \Gamma_f(z), \tag{1.16}$$

where $\Gamma_{12:odd}^{\infty}(z)$ is the odd part of function $\Gamma_{12}^{\infty}(z)$. Multiplying (1.16) by z^2, integrating it term-by-term over all z, and taking into account equality (1.15), we get that

$$\int z\Gamma_{12:odd}^{\infty}(z)\,\mathrm{d}z = -2\pi\frac{\mathrm{d}^2}{\mathrm{d}\kappa^2}G_f(\kappa)\Big|_{\kappa=0}, \tag{1.17}$$

where the spatial spectral density of the force $G_f(\kappa)$ was defined above. Since, for a Gaussian field, necessarily $\Gamma_{12:odd}^{\infty}(z) \equiv 0$, formula (1.17) implies the following proposition.

(iv) *For the existence of a Gaussian ergodic stationary regime in forced Burgers' turbulence it is necessary that*

$$G_f(\kappa) = o(\kappa^2), \quad (\kappa \to 0).$$

Hence, from (1.16), we obtain another result.

(v) *If a stationary regime in forced Burgers' turbulence is Gaussian then its spectral density satisfies condition*

$$G^{\infty}(\kappa) = \frac{1}{2\mu}\frac{G_f(\kappa)}{\kappa^2}. \tag{1.18}$$

Additional problems related to the energy dissipation mechanism in the inviscid limit ($\mu \to 0+$) and steady-state Burgers' turbulence are addressed in Appendix B. Also, in the inviscid limit, we have another result which follows from (1.18).

(vi) *If an ergodic stationary regime exists for inviscid forced Burgers' turbulence then it is non-Gaussian.*

In a sense, a solution of Burgers' equation in the inviscid limit gives a good mathematical illustration of the Cheshire cat's mystery: the viscosity disappears ($\mu = 0$) but the dissipation remains. Perhaps this is one of the reasons the study of the inviscid limit keeps attracting researchers. The subtlety of the situation is that in the inviscid limit the solutions of Burgers' equation exist only in the generalized sense. Thus, immediately one runs into the problem of selection of a suitable class of 'physically meaningful' generalized solutions and the question of uniqueness of these solutions in the selected class. Although the Hopf–Cole substitution

$$\boldsymbol{v}(\boldsymbol{x}, t) = -2\mu\nabla \ln \phi(\boldsymbol{x}, t) \tag{1.19}$$

transforms the *nonlinear* Burgers' equation (1.1) into a *linear* Schrödinger-type diffusion equation

$$\frac{\partial \phi}{\partial t} = \mu\Delta\phi - \frac{1}{2\mu}U(\boldsymbol{x}, t)\phi, \tag{1.20}$$

the inverse passage from its solutions back to the solutions of Burgers' equation in the inviscid limit is a formidable mathematical problem. The difficulty of the rigorous mathematical analysis of forced Burgers' turbulence becomes more acute if the forces are assumed to be delta-correlated in time, as we did in (1.7). In that case, equation (1.20), even for $\mu > 0$, loses its classical meaning and has to be considered as a stochastic partial differential equation in a generalized sense, like Ito or Stratonovich ordinary stochastic differential equations (see e.g. Da Prato & Zabczyk 1992). Similar difficulties arise with the definition of nonlinear terms in the original Burgers' equation (1.1). These and other mathematical problems of the theory of the forced Burgers' equation in the inviscid limit were recently actively studied in the mathematical litarature (see e.g. Bertini, Cancrini & Jona Lasinio 1994; Holden *et al.* 1994; Nakazawa 1980, and the references quoted therein). In parallel, there appeared papers written on the physical level of rigorousness, where on the basis of physical assumptions and approximations, probabilistic and spectral-correlational properties of both forced and unforced Burgers' turbulence were studied (see e.g. Tatsumi & Kida 1972; Gurbatov *et al.* 1991; Yakushkin 1981; Gurbatov & Saichev 1993; Saichev & Woyczynski 1996*b*, *c*).

In the present paper, in the inviscid limit and for an arbitrary spatial dimension $d \geqslant 1$, we provide an approximate method of analysis of the statistical properties of Burgers' turbulence for a random force field potential which is *delta-correlated in time and smooth in space*. Special attention is paid to verifying the feasibility of stationary regimes. Thus, in §2, we provide a detailed exposition of the structure of solutions of forced inviscid Burgers' turbulence (1.1). In this case the solution velocity field corresponds to the 'least-action stream' velocity among the multivalued velocity fields describing the multistream motion of non-interacting particles in the force field.

Section 3 is devoted to the statistical description of the above-mentioned multistream particle motion and proposes an approximate method of finding the desired statistical properties of Burgers' turbulence based on searching for the *largest value of least-action*. In §4, we verify the efficacy of this method for the relatively simple case of the homogeneous Burgers' equation with random initial conditions (see also Funaki, Surgailis & Woyczynski 1995 and Molchanov, Surgailis & Woyczynski 1996 for other rigorous results in this direction). We show that the results of these calculations agree well with results obtained earlier by other methods.

In §5, we study statistical properties of Burgers' turbulence and, in particular, its average kinetic energy, under the assumption that the force field has a random potential which is homogeneous in space and delta-correlated in time (a curious maximum principle for the mean kinetic energy in one-dimensional Burgers' turbulence was discovered in Hu & Woyczynski 1994, 1995). At the same time, we also explain the important role that the average stream number in the associated gas of non-interacting particles plays in the analysis of Burgers' turbulence. It turns out that the exponential growth in time of the average number of streams is a necessary condition for the existence of a stationary regime. In §6, using the theory of Markov processes, we show that the average number of streams indeed grows exponentially, at least in the one-dimensional case.

Finally, we would like to mention that in the case of simple degenerate random potentials, a rigorous analysis of the forced Burgers' turbulence has been done by Molchanov, Surgailis & Woyczynski (1995*a, b*) using the variational method and spectral analysis for the Schrödinger equation with a random potential. A statistical analysis of computer simulations for related passive tracer flows has been carried out in Janicki, Surgailis & Woyczynski (1995).

2. Least-action principle for forced Burgers' turbulence

Despite the existence of a vast literature on the subject of Burgers' turbulence, the physical meaning of inviscid forced solutions have not been studied in any detail until recently. The rigorous analysis of the problem also encounters certain difficulties (see. e.g. Holden *et al.* 1994). For this reason, we devote the present section to a rather detailed discussion of solutions of the non-homogeneous Burgers' equation (1.1) with the potential force (1.4) in an arbitrary d-dimensional space ($\boldsymbol{x} \in \mathbf{R}^d$, $d \geqslant 1$).

For the sake of simplicity we will assume in this section that the potential $U(\boldsymbol{x}, t)$ is a sufficiently smooth function in both the space variable $\boldsymbol{x}$ and time variable t. Additionally, we will complement equation (1.1) by the zero initial condition

$$\boldsymbol{v}(\boldsymbol{x}, t = 0) = 0. \tag{2.1}$$

Non-zero initial conditions can be taken into account by a special choice of the external force potential $U(\boldsymbol{x}, t)$.

As we observed in §1, by the Hopf–Cole transformation (1.19), equation (1.1) with the initial condition (2.1) is reduced to a linear Schrödinger-type diffusion equation (1.20) with initial condition

$$\varphi(\boldsymbol{x}, t = 0) = 1. \tag{2.2}$$

Its solution can be written out in the form of the well-known Feynman–Kac formula

$$\phi(\boldsymbol{x}, t) = \boldsymbol{E} \exp\left(-\frac{1}{2\mu} \int_0^t U\Big(\boldsymbol{x} - \boldsymbol{w}(t) + \boldsymbol{w}(\tau), \tau\Big)\, \mathrm{d}\tau \right), \tag{2.3}$$

where the averaging $\boldsymbol{E}$ is with respect to the ensemble of realizations of the vector-valued Wiener process $\boldsymbol{w}(t) = (w_l(t))$ whose statistical properties are determined by conditions $\boldsymbol{w}(0) = 0$, $\langle w_l(t) w_m(t)\rangle = 2\mu t \delta_{lm}$, $l, m = 1, 2, \ldots, d$, (see e.g. Carmona & Lacroix 1990 for a formal derivation).

To make the further analysis more transparent, let us write (2.3) in the form of a paths integral. For this purpose, consider a discretized form

$$U(\boldsymbol{x}, t) = \varepsilon \sum_{p=0}^{\infty} U(\boldsymbol{x}, p\varepsilon)\delta(t - p\varepsilon) \tag{2.4}$$

of the external force potential (1.4). Substituting it into (2.3) and assuming, for simplicity, that the time $t = (q+1)\varepsilon - 0$, $q = 0, 1, 2, \ldots$, is also discrete, we obtain that

$$\phi(\boldsymbol{x}, t) = \boldsymbol{E} \exp\left[-\frac{\varepsilon}{2\mu}\sum_{p=0}^{q} U\Big(\boldsymbol{x} - \sum_{r=p}^{q} \boldsymbol{\Omega}_r, p\varepsilon\Big)\right], \tag{2.5}$$

where

$$\boldsymbol{\Omega}_r = \boldsymbol{w}((r+1)\varepsilon) - \boldsymbol{w}(r\varepsilon), \qquad r = 0, 1, 2, \ldots,$$

are mutually independent Gaussian random vectors with the correlation tensor

$$\langle \Omega_{rl}\Omega_{rm}\rangle = 2\mu\varepsilon\delta_{lm}, \qquad l, m = 1, 2, \ldots, d.$$

Writing explicitly the average in (2.5) with respect to the Gaussian ensemble $\{\boldsymbol{\Omega}_0, \boldsymbol{\Omega}_2, \ldots, \boldsymbol{\Omega}_q\}$ we get

$$\phi(\boldsymbol{x}, t) = \int \ldots \int \exp\left[-\frac{1}{2\mu}\left(\varepsilon \sum_{p=0}^{q} U\Big(\boldsymbol{x} - \sum_{r=p}^{q} \boldsymbol{z}_r, p\varepsilon\Big) + \sum_{p=0}^{q} \frac{\boldsymbol{z}_p^2}{2\varepsilon}\right)\right] \mathscr{D}_{q+1}(\boldsymbol{z}), \tag{2.6}$$

where each of the above integrals denotes integration over the d-dimensional space and

$$\mathscr{D}_{q+1}(\boldsymbol{z}) = \left(\frac{1}{4\pi\mu\varepsilon}\right)^{d(q+1)/2} \mathrm{d}^d z_0\, \mathrm{d}^d z_1 \ldots \mathrm{d}^d z_q. \tag{2.7}$$

Remember that our final goal is to find not the auxiliary field $\phi(\boldsymbol{x}, t)$ but the solution $\boldsymbol{v}(\boldsymbol{x}, t)$ of the non-homogeneous Burgers' equation (1.1), expressed through the former via the Hopf–Cole formula (1.19). In that solution, in addition to $\phi(\boldsymbol{x}, t)$ itself, there also appears its gradient which we shall find by acting with the operator ∇ on the right-hand side of equality (2.6). Putting the derivatives under the integral signs, noticing that

$$\frac{\partial}{\partial x_l}\exp\left[-\frac{\varepsilon}{2\mu}\sum_{p=0}^{q} U\Big(\boldsymbol{x} - \sum_{r=p}^{q} \boldsymbol{z}_r, p\varepsilon\Big)\right] = -\frac{\partial}{\partial z_{ql}}\exp\left[-\frac{\varepsilon}{2\mu}\sum_{p=0}^{q} U\Big(\boldsymbol{x} - \sum_{r=p}^{q} \boldsymbol{z}_r, p\varepsilon\Big)\right]$$

and integrating by parts the integral with respect to $\boldsymbol{z}_q$, we obtain that

$$-2\mu\nabla\phi(\boldsymbol{x}, t) = \int \ldots \int \frac{\boldsymbol{z}_q}{\varepsilon} \exp\left[-\frac{\varepsilon}{2\mu}\left(\sum_{p=0}^{q} U\Big(\boldsymbol{x} - \sum_{r=p}^{q} \boldsymbol{z}_r, p\varepsilon\Big) + \frac{1}{2}\Big(\frac{\boldsymbol{z}_p}{\varepsilon}\Big)^2\right)\right] \mathscr{D}_{q+1}(\boldsymbol{z}). \tag{2.8}$$

Let us change the variables in integrals (2.6) and (2.8) from $\{\boldsymbol{z}_p\}$ to

$$\boldsymbol{X}_p = \boldsymbol{x} - \sum_{r=p}^{q} \boldsymbol{z}_r, \qquad p = 0, 1, \ldots, q, \qquad \boldsymbol{X}_{q+1} = x,$$

so that $\boldsymbol{z}_p = \boldsymbol{X}_{p+1} - \boldsymbol{X}_p$, $p = 0, 1, \ldots, q$, and, as a result, equalities (2.6) and (2.8) take the form

$$\phi(\boldsymbol{x}, t) = \int \ldots \int \exp\left[-\frac{\varepsilon}{2\mu}\sum_{p=0}^{q}\left(U\left(\boldsymbol{X}_p, p\varepsilon\right) + \frac{1}{2}\left(\frac{\boldsymbol{X}_{p+1} - \boldsymbol{X}_p}{\varepsilon}\right)^2\right)\right]\mathscr{D}_{q+1}(\boldsymbol{X}), \quad (2.9)$$

$$-2\mu\nabla\phi(\boldsymbol{x}, t) =$$

$$\int \ldots \int \frac{\boldsymbol{x} - \boldsymbol{X}_q}{\varepsilon}\exp\left[-\frac{\varepsilon}{2\mu}\sum_{p=0}^{q}\left(U\left(\boldsymbol{X}_p, p\varepsilon\right) + \frac{1}{2}\left(\frac{\boldsymbol{X}_{p+1} - \boldsymbol{X}_p}{\varepsilon}\right)^2\right)\right]\mathscr{D}_{q+1}(\boldsymbol{X}). \quad (2.10)$$

Let us pass in the formulas (2.9) to the limit

$$\varepsilon \to 0, \qquad q = (t - \varepsilon)/\varepsilon \to \infty.$$

Note that $\boldsymbol{X}_p$ can be naturally regarded as the values, for $\tau = p\varepsilon$, of a certain vector-valued process $\boldsymbol{X}(\tau)$: $\boldsymbol{X}_p = \boldsymbol{X}(p\varepsilon)$, so that the multiple integrals (2.9) can be interpreted as discretized functional integrals

$$\phi(\boldsymbol{x}, t) = \int \exp\left(-\frac{1}{2\mu}S[\boldsymbol{X}(\tau)]\right)\mathscr{D}[\boldsymbol{X}(\tau)], \quad (2.11)$$

$$-2\mu\nabla\phi(\boldsymbol{x}, t) = \int \frac{\mathrm{d}\boldsymbol{X}(\tau)}{\mathrm{d}\tau}\bigg|_{\tau=t}\exp\left(-\frac{1}{2\mu}S[\boldsymbol{X}(\tau)]\right)\mathscr{D}[\boldsymbol{X}(\tau)], \quad (2.12)$$

over all the sample paths $\boldsymbol{X}(\tau), \tau \in [0, t]$, satisfying the obvious condition

$$\boldsymbol{X}(\tau = t) = \boldsymbol{x}. \quad (2.13)$$

In (2.11), the *action functional* appears:

$$S[\boldsymbol{X}(\tau)] = \int_0^t \left[U(\boldsymbol{X}(\tau), \tau) + \frac{1}{2}\left(\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}\tau}\right)^2\right]\mathrm{d}\tau. \quad (2.14)$$

Substituting (2.11) in the Hopf–Cole formula (1.19), we obtain a solution of the non-homogeneous Burgers' equation (1.1), expressed through the functional integrals

$$\boldsymbol{v}(\boldsymbol{x}, t) = \frac{\displaystyle\int \frac{\mathrm{d}\boldsymbol{X}(\tau)}{\mathrm{d}\tau}\Big|_{\tau=t}\exp\left(-\frac{1}{2\mu}S[\boldsymbol{X}(\tau)]\right)\mathscr{D}[\boldsymbol{X}(\tau)]}{\displaystyle\int \exp\left(-\frac{1}{2\mu}S[\boldsymbol{X}(\tau)]\right)\mathscr{D}[\boldsymbol{X}(\tau)]}. \quad (2.15)$$

For arbitrary $\mu > 0$, the above functional form of the non-homogeneous Burgers' equation's solution is poorly suited for analytic calculations. Nevertheless, for $\mu \to 0+$, expression (2.15) supplies a geometrically helpful Lagrangian picture of the corresponding generalized solution which is an analogue of Feynman's least-action principle in quantum electrodynamics.

Least-action principle for forced Burgers' turbulence. *In the inviscid limit,*

$$\boldsymbol{v}(\boldsymbol{x}, t) = \frac{\mathrm{d}\boldsymbol{X}(\tau)}{\mathrm{d}\tau}\bigg|_{\tau=t}, \quad (2.16)$$

where $\boldsymbol{X}(\tau)$ is the vector-valued process on which the action functional (2.14) takes the minimal absolute value.

Note that analogous constructions of generalized solutions of first-order nonlinear partial differential equations can be found in the mathematical literature (see e.g. Oleinik 1957, for the one-dimensional case, and Lions 1982, for the multidimensional case).

The extremals of functional (2.14) fulfil equations

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}\tau} = \boldsymbol{V}, \qquad \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}\tau} = \boldsymbol{f}(\boldsymbol{X}, \tau), \tag{2.17}$$

together with boundary condition (2.13) combined with another obvious condition at $\tau = 0$:

$$\boldsymbol{V}(\tau = 0) = 0, \qquad \boldsymbol{X}(\tau = t) = \boldsymbol{x}. \tag{2.18}$$

Equations (2.17), along with

$$\frac{\mathrm{d}S}{\mathrm{d}t} = U(\boldsymbol{X}, \tau) + \tfrac{1}{2}\boldsymbol{V}^2, \tag{2.19}$$

$$S(\tau = 0) = 0$$

for the action functional, form a system of characteristic equations corresponding to the following first-order p.d.e.'s with respect to the field $S(\boldsymbol{x}, t)$ and its gradient $\boldsymbol{v}(\boldsymbol{x}, t) = \nabla S(\boldsymbol{x}, t)$:

$$\frac{\partial S}{\partial t} + \tfrac{1}{2}(\nabla S)^2 = U(\boldsymbol{x}, t), \tag{2.20}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \boldsymbol{f}(\boldsymbol{x}, t). \tag{2.21}$$

The latter have a clear-cut physical meaning as they describe the action and the velocity fields for a gas of non-interacting particles in the hydrodynamic limit.

If the external force $\boldsymbol{f}(\boldsymbol{x}, t)$ is a sufficiently smooth function of its arguments, then there exists a

$$t_1 > 0,$$

such that for $0 < t < t_1$ the solutions of equations (2.20) and (2.21) exist, are unique and continuous for any $\boldsymbol{x} \in \mathbf{R}^d$. At this initial stage, until the formation of discontinuities in the profile of generalized solution (2.16), it coincides with the solution of equation (2.21).

For $t > t_1$, the boundary-value problem (2.17)–(2.19) may, for some $\boldsymbol{x}$, have $N > 1$ solutions

$$\{\boldsymbol{X}_m(\tau), \boldsymbol{V}_m(\tau), S_m(\tau), m = 1, 2, \ldots, N\}. \tag{2.22}$$

Its values for $\tau = t$ and given m,

$$\boldsymbol{v}_m(\boldsymbol{x}, t) = \boldsymbol{V}_m(\tau = t), \qquad S_m(\boldsymbol{x}, t) = S_m(\tau = t),$$

can be conveniently thought of as values of a multistream solution of equations (2.20),(2.21) in the mth stream. Let us enumerate the streams in the increasing order

$$S_1(\boldsymbol{x}, t) < S_2(\boldsymbol{x}, t) < \ldots < S_N(\boldsymbol{x}, t). \tag{2.23}$$

Then the generalized solution (2.16), taking into account the appearance of discontinuities, can be written in the form

$$\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{v}_1(\boldsymbol{x}, t), \tag{2.24}$$

(see figure 1).

The above discussion can be summarized by the following statement which forms the basis of the remainder of this paper:

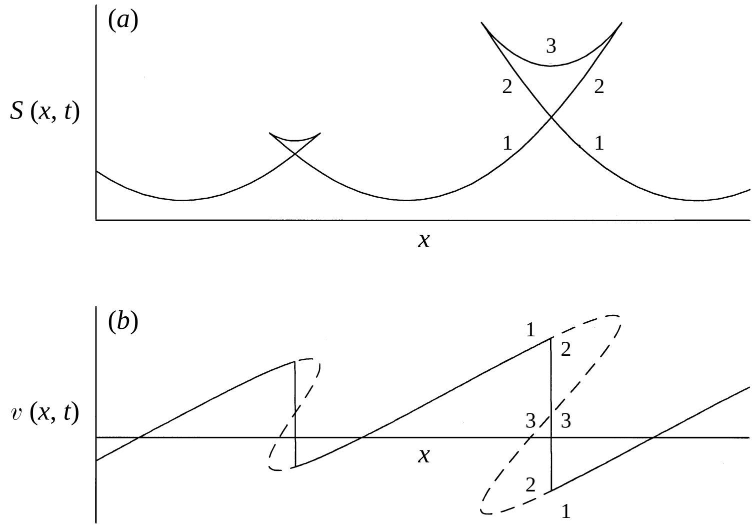

FIGURE 1. A one-dimensional example of multistream fields of action $S(x,t)$ (*a*), and of velocity $v(x,t)$ (*b*). The solid line in (*b*) indicates the stream that corresponds to the generalized solution of the inviscid Burgers' equation. The numerals give the stream numbers.

Conclusion. *The physically significant inviscid limit solutions of the non-homogeneous Burgers' equation are fully determined by multistream properties of the gas of non-interacting particles.*

3. Forced inviscid Burgers' turbulence and the multistream regimes

In this section we carry out a statistical analysis of solutions of the randomly forced Burgers' equation. In what follows we shall assume that the potential field $U(\boldsymbol{x},t)$ of the random force $\boldsymbol{f}(\boldsymbol{x},t)$ is a Gaussian random field statistically homogeneous and isotropic in space, and delta-correlated in time, with zero mean and correlation function

$$\langle U(\boldsymbol{x},t)U(\boldsymbol{x}+\boldsymbol{y},t+\theta)\rangle = 2a(y)\delta(\theta). \tag{3.1}$$

Therefore, the random force $\boldsymbol{f}(\boldsymbol{x},t)$ is also a Gaussian field statistically isotropic in space with correlation tensor

$$\Big\langle f_l(\boldsymbol{x},t)f_m(\boldsymbol{x}+\boldsymbol{y},t+\theta)\Big\rangle = 2\delta(\theta)\left[b(y)\delta_{lm} + \frac{y_l y_m}{y}\frac{\mathrm{d}b(y)}{\mathrm{d}y}\right], \tag{3.2}$$

where

$$b(y) = -\frac{1}{y}\frac{\mathrm{d}a(y)}{\mathrm{d}y}, \quad l,m = 1,2,\ldots,d.$$

3.1. *Statistical description of the auxiliary multistreams*

As we observed before, the statistical analysis of inviscid Burgers' turbulence reduces to the statistical analysis of the stochastic boundary-value problem (2.17)–(2.19). The presence of boundary conditions, even for the random force $\boldsymbol{f}$ delta-correlated in time, does not permit direct use of the Markov processes apparatus in the analysis of statistics of solutions (2.17)–(2.19). To make those powerful tools available, one has to formulate initially an auxiliary Cauchy problem, the statistical properties of which will determine the desired statistical properties of the boundary-value problem (2.17)–(2.19). As it will become clear from what follows, it is natural to take as such an auxiliary problem the Cauchy problem

$$\left.\begin{aligned} \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{V}, \qquad \frac{\mathrm{d}S}{\mathrm{d}t} = U(\boldsymbol{X},t) + \tfrac{1}{2}V^2, \qquad \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{X},t), \\ \boldsymbol{X}(\boldsymbol{y},t=0) = \boldsymbol{y}, \qquad S(\boldsymbol{y},t=0) = \boldsymbol{V}(\boldsymbol{y},t=0) = 0, \end{aligned}\right\} \tag{3.3}$$

$$\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = \boldsymbol{K}, \quad \frac{\mathrm{d}\boldsymbol{K}}{\mathrm{d}t} = \boldsymbol{g}(X,t)\boldsymbol{J}, \qquad \boldsymbol{J}(\boldsymbol{y},t=0) = \boldsymbol{I}, \qquad \boldsymbol{K}(\boldsymbol{y},t=0) = 0, \tag{3.4}$$

for the scalar field $S(\boldsymbol{y},t)$, vector fields $\boldsymbol{X}(\boldsymbol{y},t)$ and $\boldsymbol{V}(\boldsymbol{y},t)$, and also tensor fields $\boldsymbol{J}(\boldsymbol{y},t)$ and $\boldsymbol{K}(\boldsymbol{y},t)$ with components

$$J_{lm}(\boldsymbol{y},t) = \frac{\partial X_l}{\partial y_m}, \qquad K_{lm} = \frac{\partial V_l}{\partial y_m}.$$

The following notation has been used: $\hat{I}$ is the diagonal unit matrix and $\hat{g}$ is a random tensor with components

$$g_{lm}(\boldsymbol{x},t) = \frac{\partial^2 U(\boldsymbol{x},t)}{\partial x_l \partial x_m}. \tag{3.5}$$

The Cauchy problem (3.3) has a clear-cut intuitive physical interpretation. It describes the evolution of coordinates $\boldsymbol{X}$, action S, and velocity $\boldsymbol{V}$ of particles forced by $\boldsymbol{f}(\boldsymbol{x},t)$. The notation clearly displays the dependence on the initial coordinates $\boldsymbol{y}$ of the particle. This dependence plays a fundamental role in the further analysis. The Cauchy problems (3.3) and (3.4) together with arbitrarily distributed initial positions $\boldsymbol{y}$ can be naturally interpreted as a gas of non-interacting particles. The tensors $\boldsymbol{J}$ and $\boldsymbol{K}$ describe the deformation of an infinitesimal volume 'frozen' in the gas. Recall that $\boldsymbol{y}$ are Lagrangian coordinates of this gas. Their connection with the Eulerian coordinates $\boldsymbol{x}$ is given by a vector equality

$$\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{y},t). \tag{3.6}$$

For given $\boldsymbol{x}$ and t it is an equation with respect to $\boldsymbol{y}$. Solving it, we obtain

$$\boldsymbol{y} = \boldsymbol{Y}(\boldsymbol{x},t), \tag{3.7}$$

the Lagrangian coordinates of particles which at time t arrive at a point with Eulerian coordinates $\boldsymbol{x}$. We should emphasize that in the general case, the gas of non-interacting particles has several, say $N(\boldsymbol{x},t) \geqslant 1$, streams. It means that equation (3.6) may have several roots. In this case, equation (3.7) defines a multi-valued function assuming N values

$$\boldsymbol{Y}_1(\boldsymbol{x},t), \quad \boldsymbol{Y}_2(\boldsymbol{x},t), \ \ldots\ , \boldsymbol{Y}_N(\boldsymbol{x},t). \tag{3.8}$$

Consider the joint probability density of the solutions of the auxiliary Cauchy problem (3.3)–(3.4):

$$\begin{aligned} &\mathscr{P}(\boldsymbol{x},s,\boldsymbol{v},\boldsymbol{j},\boldsymbol{\kappa};\boldsymbol{y},t) \\ &\quad = \Big\langle \delta(\boldsymbol{X}(\boldsymbol{y},t)-\boldsymbol{x})\delta(S(\boldsymbol{y},t)-s)\delta(\boldsymbol{V}(\boldsymbol{y},t)-\boldsymbol{v})\delta(\boldsymbol{J}(\boldsymbol{y},t)-\boldsymbol{j})\delta(\boldsymbol{K}(\boldsymbol{y},t)-\boldsymbol{\kappa})\Big\rangle. \end{aligned} \tag{3.9}$$

Let us transform the right-hand side of equality (3.9), using the well-known identity

$$\delta(\boldsymbol{x}-\boldsymbol{X}(\boldsymbol{y},t)) = \sum_{n=1}^{N(\boldsymbol{x},t)} \frac{\delta(\boldsymbol{Y}_n(\boldsymbol{x},t)-\boldsymbol{y})}{|J(\boldsymbol{Y}_n,t)|}, \tag{3.10}$$

for the delta-function (see e.g. Saichev & Woyczynski 1996*a*), where

$$J(\boldsymbol{y},t) = \|\boldsymbol{J}(\boldsymbol{y},t)\| = \left\| \frac{\partial X_l}{\partial y_m} \right\|, \tag{3.11}$$

is the Jacobian of the Eulerian-to-Lagrangian coordinate transformation. Substituting (3.10) into (3.9) and taking into account the probing property of the delta-function, we have

$$|j|\mathscr{P}(\boldsymbol{x},s,\boldsymbol{v},\boldsymbol{j},\boldsymbol{l};\boldsymbol{y},t) =$$

$$\left\langle \sum_{n=1}^{N(\boldsymbol{x},t)} \delta(\boldsymbol{Y}_n(\boldsymbol{x},t)-\boldsymbol{y})\delta(s_n(\boldsymbol{x},t)-s)\delta(\boldsymbol{v}_n(\boldsymbol{x},t)-\boldsymbol{v})\delta(\boldsymbol{j}_n(\boldsymbol{x},t)-\boldsymbol{j})\delta(\boldsymbol{\kappa}_n(\boldsymbol{x},t)-\boldsymbol{\kappa}) \right\rangle, \tag{3.12}$$

where

$$s_n(\boldsymbol{x},t) = S(\boldsymbol{Y}_n,t), \quad \boldsymbol{v}_n(\boldsymbol{x},t) = \boldsymbol{V}(\boldsymbol{Y}_n,t), \tag{3.13a}$$

$$\boldsymbol{j}_n(\boldsymbol{x},t) = \boldsymbol{J}(\boldsymbol{Y}_n,t), \quad \boldsymbol{\kappa}_n(\boldsymbol{x},t) = \boldsymbol{K}(\boldsymbol{Y}_n,t) \tag{3.13b}$$

are fields that describe the state of the gas in the nth of N streams which occur at point $\boldsymbol{x}$ at time t, and where j is the determinant of the matrix $\boldsymbol{j}$ ($j = \|\boldsymbol{j}\|$.) By the total probability formula, in view of (3.12),

$$|j|\mathscr{P}(\boldsymbol{x},s,\boldsymbol{v},\hat{j},\boldsymbol{\kappa};\boldsymbol{y},t) = \sum_{N=1}^{\infty} P(N;\boldsymbol{x},t) \sum_{n=1}^{N} W_n(\boldsymbol{y},s,\boldsymbol{v},\boldsymbol{j},\boldsymbol{\kappa};\boldsymbol{x},t|N), \tag{3.14}$$

where $P(N;\boldsymbol{x},t)$ is the probability of the event that at a given point $\boldsymbol{x}$ at time t we have N streams present, and where $W_n(\boldsymbol{y},s,\boldsymbol{v},\boldsymbol{j},\boldsymbol{\kappa};\boldsymbol{x},t|N)$ is the conditional joint probability density of random fields (3.8) and (3.13*a,b*) in the nth stream, given that the total number of streams is N.

3.2. *Approximations for Burgers' turbulence statistics*

In view of (2.22)–(2.24), the sought joint probability density of the least-action functional, corresponding Lagrangian coordinates $\boldsymbol{Y}(\boldsymbol{x},t)$, the generalized solution $\boldsymbol{v}(\boldsymbol{x},t)$ of the non-homogeneous Burgers equation in the inviscid limit, and the auxiliary fields $\boldsymbol{j},\boldsymbol{\kappa}$, are expressed in the following fashion through the components of sum (3.14):

$$W(\boldsymbol{y},s,\boldsymbol{v},\boldsymbol{j},\boldsymbol{\kappa};\boldsymbol{x},t) = \sum_{N=1}^{\infty} P(N;\boldsymbol{x},t) W_1(\boldsymbol{y},s,\boldsymbol{v},\boldsymbol{j},\boldsymbol{\kappa};\boldsymbol{x},t|N). \tag{3.15}$$

In the case of statistically homogeneous fields – in what follows we will restrict our attention to such fields – the probability density of the streams' number does not depend on $\boldsymbol{x}$, and the probability density in (3.14-15) depends only on $\boldsymbol{x}-\boldsymbol{y}$. Hence, integrating equalities (3.14–15) over all $\boldsymbol{x},\boldsymbol{j},\boldsymbol{\kappa}$, we arrive at the relations

$$\langle |J| \rangle_{s\boldsymbol{v}} \mathscr{P}(s,\boldsymbol{v};t) = \sum_{N=1}^{\infty} P(N;t) \sum_{n=1}^{N} W_n(s,\boldsymbol{v};t|N), \tag{3.16}$$

$$W(s,\boldsymbol{v};t) = \sum_{N=1}^{\infty} P(N;t) W_1(s,\boldsymbol{v};t|N), \tag{3.17}$$

more convenient for further analysis. Here $\langle ... \rangle_{s\boldsymbol{v}}$ denotes the average under the condition that $S(y,t) = s$, $V(\boldsymbol{y},t) = \boldsymbol{v}$ are given.

Unfortunately we cannot extract the partial sum (3.17), which is of interest to us, from the total sum (3.16). Such an operation is possible in principle, but to find (3.17) one has to have knowledge of all the joint probability densities for the Cauchy problem (3.3)–(3.4) under different initial conditions. These joint probability densities satisfy complex Kolmogorov equations whose solutions are not known. For that reason we will utilize a semi-qualitative method of finding probability densities of forced Burgers' turbulence.

Our main assumption is as follows: *there exists a number $\bar{S}(t)$ – the largest value of the least-action – such that*

$$\int_{-\infty}^{\bar{S}(t)} W_1(s;t|N)\mathrm{d}s \approx 1, \tag{3.18}$$

and

$$\int_{-\infty}^{\bar{S}(t)} W_n(s;t|N)\mathrm{d}s \approx 0, \quad n = 2, 3, \ldots N,$$

where

$$W_n(s;t|N) = \int_{-\infty}^{\infty} W_n(s, \boldsymbol{v};t|N)\mathrm{d}^d v, \quad n = 1, 2, \ldots, N.$$

If this assumption is satisfied, then the desired probability density

$$W(\boldsymbol{v};t) = \sum_{N=1}^{\infty} P(N;t) W_1(\boldsymbol{v};t|N)$$

of Burgers' turbulence can be approximated by integration of equality (3.16) over all the values of s in the interval $(-\infty, \bar{S}(t))$, that is

$$W(\boldsymbol{v};t) = \int_{-\infty}^{\bar{S}(t)} \langle |J| \rangle_{s\boldsymbol{v}} \mathscr{P}(s, \boldsymbol{v};t)\,\mathrm{d}s. \tag{3.19}$$

In addition, the value of $\bar{S}(t)$ can be determined from the normalization condition

$$1 = \int_{-\infty}^{\bar{S}(t)} \langle |J| \rangle_s \mathscr{P}(s;t)\,\mathrm{d}s \tag{3.20}$$

for probability density (3.19), where $\mathscr{P}(s;t)$ is the probability density of random action $S(\boldsymbol{y}, t)$ satisfying the auxiliary Cauchy problem (3.3).

Closing this section we will make an additional assumption that the *random Jacobian field J (3.11) is statistically independent of the random fields $S(\boldsymbol{y}, t)$ and $\boldsymbol{V}(\boldsymbol{y}, t)$.* In such a case, the expressions (3.19) for the solutions of the non-homogeneous Burgers equation and equation for the maximal value of absolute minima $\bar{\boldsymbol{S}}(t)$ (3.20) take a particularly simple form

$$W(\boldsymbol{v};t) = \langle N(t) \rangle \int_{-\infty}^{\bar{S}(t)} \mathscr{P}(s, \boldsymbol{v};t)\,\mathrm{d}s, \tag{3.21}$$

$$\langle N(t) \rangle \int_{-\infty}^{\bar{S}(t)} \mathscr{P}(s;t)\,\mathrm{d}s = 1.$$

Note that the last assumption is not really essential and has only a technical nature. If it is not satisfied then the following calculations do not change qualitatively, but they do get more complicated. In the test case considered in the next section we will verify that the statistical dependence between J and $S, \boldsymbol{V}$ does not significantly affect the final outcome. For that reason, in the remainder of this paper we will always assume J to be statistically independent of the values of the vector (S, V) and use expression (3.21) instead of the more correct, but much more complex formulas (3.19)–(3.20).

3.3. *A model example*

Let us test the conjecture underlying formulas (3.19)–(3.20) on the following simple model which, however, is relatively close to the problem we are considering. Let, for a given number of streams N, the values of actions of different streams $\{S_1, \dots S_N\}$ form a family of statistically independent random variables with identical cumulative distribution functions

$$F(s) = P(S_n < s).$$

In each realization, as in (2.23), we will form an order statistic

$$S^1 \leqslant S^2 \leqslant \dots \leqslant S^N,$$

and denote the cumulative distribution function of the nth ordered variable S^n by

$$F_n^N(s) = P(S^n < s)).$$

It is well known that

$$F_n^N(s) = 1 - \sum_{l=0}^{n-1} \binom{N}{l} \left(\frac{z}{N}\right)^l \left(1 - \frac{z}{N}\right)^{N-l}, \tag{3.22}$$

and in particular, that the cumulative distribution of the smallest $S_{min} = S^1$ is equal to

$$F_1^N(s) = P(S^1 < s) = 1 - \left(1 - \frac{z}{N}\right)^N, \tag{3.23}$$

where $z = z(s) = NF(s)$. Besides, it is clear that

$$\sum_{n=1}^{N} F_n^N = NF = z. \tag{3.24}$$

Within the framework of this example, the conditional normalizations (3.20) defining values of $\bar{S}$, reduce to the equality

$$NF = z = 1.$$

In addition, according to our assumption, conditions

$$F_1^N \Big|_{z=1} \approx 1, \quad \sum_{n=2}^{N} F_n^N \Big|_{z=1} \approx 0, \tag{3.25}$$

analogous to (3.18) have to be fulfilled. Let us verify to what extent they are valid, substituting here corresponding expressions from (3.22)–(3.24). This gives

$$\left.\begin{aligned} F_1^N(\bar{s}) = F_1^N \Big|_{z=1} &= 1 - \left(1 - \frac{1}{N}\right)^N, \\ R^N(\bar{S}) = \sum_{n=2}^{N} F_n^N \Big|_{z=1} &= \left(1 - \frac{1}{N}\right)^N. \end{aligned}\right\} \tag{3.26}$$

Now, for example, it follows that the values of $\{F_1^N(\bar{S})\}$ form a monotonically

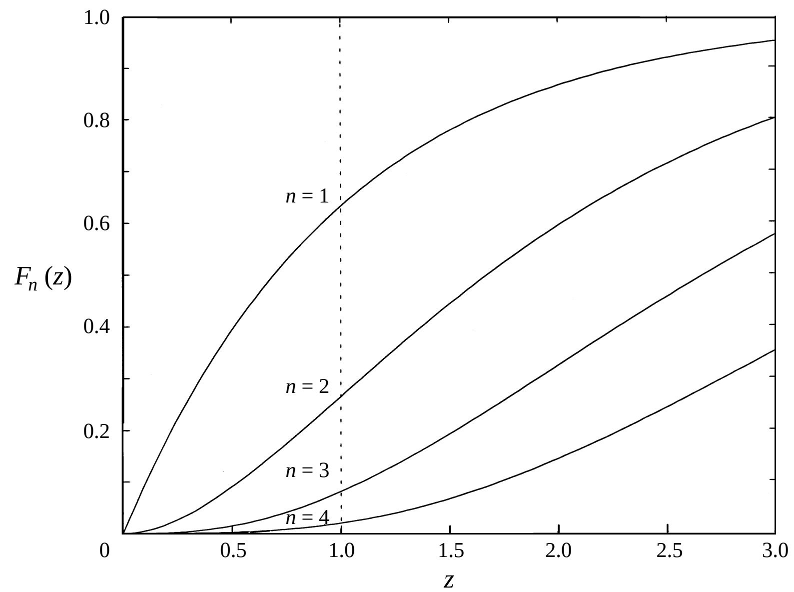

FIGURE 2. Graphs of cumulative distributions of order statistics $F_n^{\infty}(z)$. The curves have a universal form and do not depend on the distributions of original random variables $S_1, S_2, \ldots$ One can see that, for $z = 1$, the value of $F_1^{\infty}(1)$ is rather close to 1, while other cumulative distributions are still pretty small.

decreasing (as N increases) sequence, the limit of which is

$$F_1^{\infty}(z = 1) = \lim_{N\to\infty} F_1^N(\bar{s}) = 1 - \mathrm{e}^{-1}, \tag{3.27}$$

a quantity one can think of, in this context, as being 'close' to 1. Correspondingly, $\{R^N(\bar{s})\}$ is a monotonically increasing sequence which, for $N \to \infty$, converges to a 'small' number

$$R^{\infty}(z = 1) = \lim_{N\to\infty} R^N(\bar{s}) = \mathrm{e}^{-1}. \tag{3.28}$$

In this fashion, in the model example under consideration, relations (3.25) are satisfied with a 'good' accuracy, expressed by the limit equalities (3.27)–(3.28).

Furthermore, notice that for arbitrary z and large $N \to \infty$, probability densities of the order statistics $\{S^1, S^2, \ldots\}$ are described (see figure 2) by the main asymptotics of expressions (3.22)–(3.24):

$$\left.\begin{aligned} F_1^{\infty}(z) &= 1 - \mathrm{e}^{-z}, \quad F_n^{\infty}(z) = 1 - \mathrm{e}^{-z}\sum_{l=0}^{n-1}\frac{z^l}{l!}, \\ R^{\infty}(z) &= \sum_{m=2}^{\infty} F_n^{\infty}(z) = z - 1 + \mathrm{e}^{-z}. \end{aligned}\right\} \tag{3.29}$$

We should emphasize here that the original problem of finding statistical properties of solutions of the non-homogeneous Burgers' equation is related to the situation discussed in the above example at very large times, when the average number of streams

$$\langle N(t)\rangle = \sum_{m=1}^{\infty} NP(N;t) = \langle |J|\rangle \tag{3.30}$$

is much larger than 1. Indeed, for $\langle N \rangle \gg 1$ the law-of-large-numbers effects take over, the random number $N(t)$ of streams is not much different from the mean number

$$(\langle N - \langle N \rangle \rangle^2)^{1/2} \ll \langle N \rangle,$$

and one can assume that, for $\langle N \rangle \gg 1$, the number of streams in each realization is the same and equal to $\langle N(t) \rangle$.

In addition, it is natural to assume that in the multistream regime $\langle N \rangle \gg 1$, the particles which arrive at a given time at point $\boldsymbol{x}$ move along strongly dispersed paths, so that the forces acting on different particles $\boldsymbol{f}(\boldsymbol{X}_m(\tau), \tau); \tau \in [0, t]$, actually are statistically independent. Therefore, the values $\{S_1(\boldsymbol{x}, t), S_2(\boldsymbol{x}, t), \ldots, S_{\langle N \rangle}(\boldsymbol{x}, t)\}$ of their actions can be treated as independent parameters of the particles.

4. A test case: two-dimensional unforced Burgers' turbulence

We shall illustrate the above general statistical approach in the relatively well understood case of homogeneous Burgers' turbulence. To be specific, we will restrict ourselves to the two-dimensional case $\boldsymbol{x} \in \mathbf{R}^2$. Then, the potential

$$U(\boldsymbol{x}, t) = S_0(\boldsymbol{x})\delta(t),$$

where $S_0(\boldsymbol{x})$ is the initial velocity field potential, that is

$$\boldsymbol{v}_0(\boldsymbol{x}) = \nabla S_0(\boldsymbol{x}).$$

Taking this into account, the auxiliary Cauchy problem (3.3)–(3.4) takes the following form:

$$\left.\begin{array}{c}
\dfrac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{V}, \quad \dfrac{\mathrm{d}S}{\mathrm{d}t} = \tfrac{1}{2}V^2, \quad \dfrac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = 0, \\
\boldsymbol{X}(\boldsymbol{y}, t=0) = \boldsymbol{y}, \quad S(\boldsymbol{y}, t=0) = S_0(\boldsymbol{y}), \quad \boldsymbol{V}(\boldsymbol{y}, t=0) = \boldsymbol{v}_0(\boldsymbol{y}), \\
\dfrac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = \boldsymbol{K}, \quad \dfrac{\mathrm{d}\boldsymbol{K}}{\mathrm{d}t} = 0, \quad \boldsymbol{J}(\boldsymbol{y}, t=0) = \boldsymbol{I}, \quad \boldsymbol{K}(\boldsymbol{y}, t=0) = \boldsymbol{K}_0(\boldsymbol{y}),
\end{array}\right\} \tag{4.1}$$

where $\boldsymbol{K}_0(\boldsymbol{x})$ is a tensor with components

$$K_{0lm}(\boldsymbol{x}) = \frac{\partial^2 S_0(\boldsymbol{x})}{\partial x_l \partial x_m}.$$

Let $S_0(\boldsymbol{x})$ be a Gaussian, statistically isotropic field with zero mean and correlation function

$$\langle S_0(\boldsymbol{x}) S_0(\boldsymbol{x} + \boldsymbol{y}) \rangle = \frac{\sigma_0^2}{\kappa^2} \exp\left(-\tfrac{1}{2}\kappa^2 y^2\right).$$

Then the fields $S_0(\boldsymbol{x})$ and $\boldsymbol{v}_0(\boldsymbol{x})$ are statistically independent at the same spatial point, and the joint probability density of solutions S and $\boldsymbol{V}$ of the Cauchy problem (4.1) takes the form

$$\mathscr{P}(s, \boldsymbol{v}; t) = w_v(\boldsymbol{v}) w_s(s - \boldsymbol{v}^2 t/2), \tag{4.2}$$

where $w_v(\boldsymbol{v})$ and $w_s(s)$ are, respectively, the probability densities of fields $\boldsymbol{v}_0(\boldsymbol{x}), S_0(\boldsymbol{x})$

which, in the two-dimensional case, are

$$w_v(\boldsymbol{v}) = \frac{1}{2\pi\sigma_0^2} \exp\left(-\frac{\boldsymbol{v}^2}{2\sigma_0^2}\right), \tag{4.3}$$

$$w_s(s) = \frac{\kappa}{(2\pi)^{1/2}\sigma_0} \exp\left(-\frac{s^2\kappa^2}{2\sigma_0^2}\right). \tag{4.4}$$

For convenience, let us introduce a dimensionless scalar field

$$u(\boldsymbol{x}, t) = \boldsymbol{v}^2(\boldsymbol{x}, t)/2\sigma_0^2. \tag{4.5}$$

It follows from (3.21) and (4.2)–(4.4) that its probability density is given by the formula

$$W(u; t) = \tfrac{1}{2}\langle N(t)\rangle \mathrm{e}^{-u}\mathrm{erfc}\,(u\tau - \rho), \tag{4.6}$$

where the quantity ρ is determined from the normalization condition

$$\langle N(t)\rangle \int_0^\infty \mathrm{e}^{-u}\mathrm{erfc}\,(u\tau - \rho)\mathrm{d}u = 2,$$

which is not difficult to transform into the following form, more convenient for our analysis:

$$\langle N(t)\rangle \left[\mathrm{erfc}\,(-\rho) - \exp\left(-\rho^2 + \left(\rho - \frac{1}{2\tau}\right)^2\right) \mathrm{erfc}\left(\frac{1}{2\tau} - \rho\right)\right] = 2. \tag{4.7}$$

In (4.6)–(4.7), we have introduced the following dimensionless variables:

$$\rho = \kappa\bar{S}/\sqrt{2}\sigma_0, \qquad \tau = \kappa\sigma_0 t/\sqrt{2}, \tag{4.8}$$

and the notation

$$\mathrm{erfc}\,(z) = 1 - \mathrm{erf}\,(z), \qquad \mathrm{erf}\,(z) = \frac{2}{\pi^{1/2}} \int_0^z \mathrm{e}^{-y^2}\mathrm{d}y \tag{4.9}$$

was used for the special error function.

Expressions (4.6)–(4.7) contain the mean value $\langle N(t)\rangle$ of the streams' number, which will be calculated below. For now, assuming that $\langle N(t)\rangle$ is known, observe that it is not very difficult to solve equation (4.7) numerically with respect to $\rho(\tau)$, and define the probability density (4.6) and corresponding moment functions for any τ. Here, we will restrict ourselves to the derivation of the asymptotic formulas for the late stage when multiple discontinuities coalesce ($\tau \gg 1, \langle N(t)\rangle \gg 1$) in Burgers' turbulence. At that stage, equation (4.7) can be replaced, with help of the asymptotic formula

$$\mathrm{erfc}\,(z) \sim \frac{1}{\pi^{1/2}z}\mathrm{e}^{-z^2}, \qquad z \to \infty, \tag{4.10}$$

by the asymptotic relation

$$\rho^2\mathrm{e}^{\rho^2} = \frac{\langle N(t)\rangle}{4\tau\pi^{1/2}}. \tag{4.11}$$

If the right-hand side of this equality is much larger than 1, then we get the following asymptotic formula:

$$|\rho| \sim \left[\ln\left(\frac{\langle N(t)\rangle}{4\tau\pi^{1/2}}\right)\right]^{1/2}, \qquad \rho < 0, \quad |\rho| \gg 1. \tag{4.12}$$

Let us substitute expression (4.12) into (4.6). Using (4.10), we arrive at the following result:

(i) *For* $\tau \gg 1$, $\langle N(t)\rangle \gg 1$, *and* $|\rho| \gg 1$, *the dimensionless kinetic energy* $u = \boldsymbol{v}^2/2\sigma_0^2$ *in unforced Burgers' turbulence has the probability density*

$$W(u;\tau) = 2|\rho|\tau \exp(-2|\rho|\tau u), \tag{4.13}$$

where ρ *and* τ *are given by (4.8).*

In particular, it follows that the average dimensionless kinetic energy $\langle u(x,t)\rangle$ in Burgers' turbulence in the late stage of multiple shock coalescence, satisifes the asymptotic law

$$\langle u(\boldsymbol{x},t)\rangle \sim 1/2|\rho|\tau. \tag{4.14}$$

In relations (4.12)–(4.14), the principal role was played by the average number $\langle N(t)\rangle$ of streams in the gas of non-interacting particles. Let us calculate that number in the two-dimensional case under consideration. For that purpose recall that this average is connected by formula (3.30) with statistical characteristics of the Jacobian $J(\boldsymbol{y},t)$ (3.11):

$$\langle N(t)\rangle = \langle |J|\rangle.$$

It is known (see e.g. Gurbatov *et al.* 1991) that in the two-dimensional case the Jacobian is statistically equivalent to the following random quantity:

$$J = (1+2\alpha)^2 - 2\beta,$$

where α, $-\infty < \alpha < \infty$, and $\beta \geqslant 0$ are statistically independent random quantities with probability densities

$$\mathscr{P}(\alpha;\tau) = \frac{1}{(2\pi)^{1/2}\tau}\exp\left(-\frac{\alpha^2}{2\tau^2}\right), \qquad \mathscr{Q}(\beta;\tau) = \frac{1}{(2\tau^2)^{1/2}}\exp\left(-\frac{\beta}{2\tau^2}\right).$$

The above two formulas permit us to obtain an exact expression for the probability density of the Jacobian:

$$\mathscr{P}(j;\tau) = \frac{1}{8\sqrt{3}\tau^2}\exp\left(\frac{j}{4\tau^2} - \frac{1}{12\tau^2}\right)$$
$$\times \begin{cases} 2 & \text{if } j<0 \\ 2 - \operatorname{erf}\left(\left(\frac{3j}{8\tau^2}\right)^{1/2} - \frac{1}{4\sqrt{3}\tau}\right) - \operatorname{erf}\left(\left(\frac{3j}{8\tau^2}\right)^{1/2} + \frac{1}{4\sqrt{3}\tau}\right) & \text{if } j>0. \end{cases} \tag{4.15}$$

wherefrom, after simple calculations, we obtain that

$$\langle N(t)\rangle = 1 + \frac{8}{\sqrt{3}}\tau^2\exp\left(-\frac{1}{12\tau^2}\right). \tag{4.16}$$

In particular, for $\tau \to \infty$, the average number of streams satisfies the following asymptotic power law:

$$\langle N(t)\rangle \sim \frac{8}{\sqrt{3}}\tau^2. \tag{4.17}$$

Subsituting it into (4.12), we find that

$$|\rho| \sim \left[\ln\left(\frac{\tau}{(3\pi)^{1/2}}\right)\right]^{1/2}, \qquad \tau \gg 1, \tag{4.18}$$

which gives the following result:

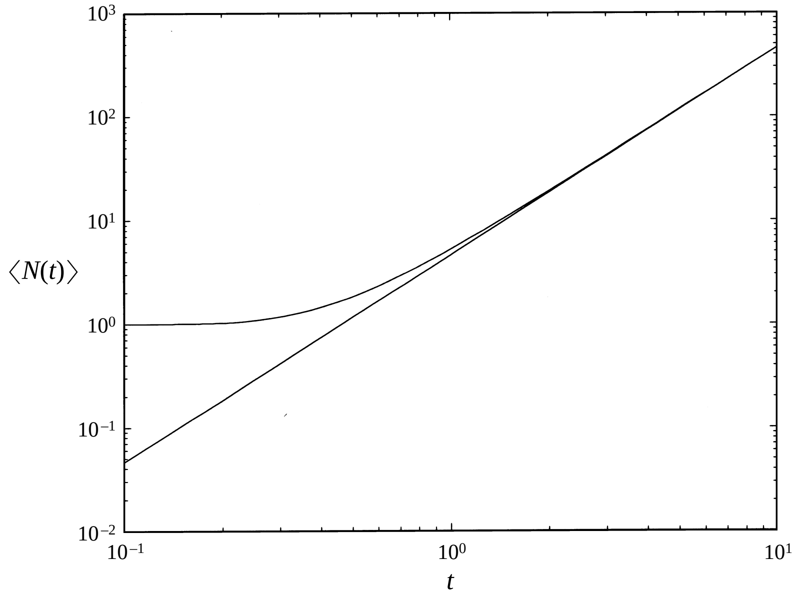

FIGURE 3. Time evolution of the exact (top line; see (4.16)) and asymptotic (bottom line; see (4.17)) average number $\langle N(t)\rangle$ of streams in two-dimensional homogeneous Burgers' turbulence. Initially, when the dimensionless time $\tau < 1$, the number of streams is close to 1, and the discontinuities of $v(x,t)$ are practically absent. In the late stages $\tau \gg 1$ of the multiple shock coalescence, the number of shocks is well described by the asymptotic formula (4.17).

(ii) *The average kinetic energy (4.14) of the unforced two-dimensional inviscid Burgers turbulence decays, at sufficiently large times, as*

$$\langle u(\boldsymbol{x},t)\rangle \sim \left(2\tau\left[\ln\left(\frac{\tau}{(3\pi)^{1/2}}\right)\right]^{1/2}\right)^{-1},$$

where $\tau = \kappa\sigma_0 t/\sqrt{2}$.

Note that the above result agrees well with the asymptotic expression for the average kinetic energy in Burgers' turbulence obtained for the one-dimensional problem by a different asymptotic approach in Tatsumi Kida (1972) and Gurbatov *et al.* (1991). This also is indirect evidence in support of our conjecture that S and V are actually statistically independent of the Jacobian J. Recall that this conjecture permitted us to replace the more precise expressions (3.19)–(3.20) by expressions (3.21) which are more convenient for calculations.

Remark 4.1. For $\tau \to \infty$, the probability density of the Jacobian (4.15) has the following *self-similar* property:

$$\mathscr{P}(j;\tau) \sim \frac{1}{c\langle N\rangle}\mathscr{P}_\infty\left(\frac{j}{c\langle N\rangle}\right), \tag{4.19}$$

where

$$\mathscr{P}_\infty\left(\frac{z}{c}\right) = \frac{2}{3}\exp\left(\frac{z}{\sqrt{12}}\right) \times \begin{cases} 1 & \text{if } z < 0 \\ 1 - \operatorname{erf}\left(\sqrt{z\sqrt{3}}\right) & \text{if } z > 0, \end{cases}$$

and c is a normalizing constant, which in this case is

$$c = \frac{4}{\sqrt{15}}\left(3\sqrt{6} - 2\sqrt{5}\right).$$

The self-similarity (4.19) of the Jacobian's probability density, which is clear in the homogeneous case, will be used later on in the multidimensional and forced case as an assumption under which we will find the rate of growth for the time evolution of $\langle N(t)\rangle$ and of the average kinetic energy. In the one-dimensional case, we will be able to use a more precise approach to study the convergence of Burgers' turbulence to a stationary regime.

5. Statistical properties of forced Burgers' turbulence

5.1. *Statistics of non-interacting particles' action*

Let us apply the algorithm proposed above to the calculation of statistical properties of forced Burgers' turbulence. First, we shall study the probability density of solutions of the auxiliary Cauchy problem (3.3). It follows from (3.1)–(3.3) that action $S(\boldsymbol{y}, t)$ can be represented as a sum of two statistically independent summands

$$S(\boldsymbol{y}, t) = S_1(\boldsymbol{y}, t) + S_2(\boldsymbol{y}, t). \tag{5.1}$$

Moreover, the first summand is also independent of the random velocity $\boldsymbol{V}(\boldsymbol{y}, t)$ and has the probability density

$$\mathscr{P}_1(s; t) = \frac{1}{2(\pi a t)^{1/2}} \exp\left(-\frac{s^2}{4at}\right), \qquad a = a(0). \tag{5.2}$$

Furthermore, the joint probability density of the second summand in (5.1) and the velocity field $\boldsymbol{V}(\boldsymbol{y}, t)$ satisfies the following Kolmogorov equation

$$\frac{\partial \mathscr{P}_2}{\partial t} + \frac{1}{2}v^2 \frac{\partial \mathscr{P}_2}{\partial s} = b\Delta_v \mathscr{P}_2, \qquad b = b(0), \tag{5.3}$$

$$\mathscr{P}_2(s, \boldsymbol{v}; t = 0) = \delta(s)\delta(\boldsymbol{v}).$$

Respectively, the joint probability density of the full action $S(\boldsymbol{y}, t)$ and the velocity $\boldsymbol{V}(\boldsymbol{y}, t)$ is equal to

$$\mathscr{P}(s, \boldsymbol{v}; t) = \mathscr{P}_1(s; t) \otimes \mathscr{P}_2(s, \boldsymbol{v}; t), \tag{5.4}$$

where the symbol $\otimes$ means the convolution operation, here with respect to variable s.

Let us pass from (5.3) to an equation for the function

$$\theta(\mu, \boldsymbol{v}; t) = \int_0^\infty \mathrm{d}s \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \mathscr{P}_2(s, \boldsymbol{v}; t) \exp[-\mu s + \mathrm{i}(\boldsymbol{v} \cdot \boldsymbol{v})]\mathrm{d}^d v. \tag{5.5}$$

That equation has the form

$$\frac{\partial \theta}{\partial t} = \frac{\mu}{2}\Delta_v \theta - bv^2\theta, \qquad \theta(\mu, \boldsymbol{v}; t = 0) = 1. \tag{5.6}$$

We shall look for a solution of this Cauchy problem in the form

$$\theta(\mu, \boldsymbol{v}; t) = \exp[q(\mu, t) - \tfrac{1}{2}p(\mu, t)\boldsymbol{v}^2]. \tag{5.7}$$

Substituting (5.7) into (5.6), we arrive at the following equations for q and p:

$$\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{\mu d}{2} p = 0, \qquad q(\mu, 0) = 0,$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} + \mu p^2 = 2b, \qquad p(\mu, 0) = 0,$$

the solutions thereof, under the initial conditions indicated above, are

$$p(\mu, t) = \frac{\tau}{\delta^{1/2}} \tanh \delta^{1/2}, \quad q(\mu, t) = -\frac{d}{2} \ln(\cosh \delta^{1/2}),$$

where new variables

$$\delta = 2\mu b t^2, \qquad \tau = 2bt, \tag{5.8}$$

have been introduced. Substituting the above expressions for p and q into (5.7), we obtain that

$$\theta(\mu, \boldsymbol{v}; t) = \left(\frac{1}{\cosh \delta^{1/2}} \right)^{d/2} \exp\left(-\frac{\tau}{2} \boldsymbol{v}^2 \frac{\tanh \delta^{1/2}}{\delta^{1/2}} \right). \tag{5.9}$$

In particular, for $\boldsymbol{v} = 0$, we have the expression

$$\theta_2(\mu; t) = \left(\frac{1}{\cosh \delta^{1/2}} \right)^{d/2} \tag{5.10}$$

for the Laplace transform

$$\theta_2(\mu; t) = \int_0^\infty \mathrm{e}^{-\mu s} \mathscr{P}_2(s; t)\, \mathrm{d}s \tag{5.11}$$

of the probability density of the second action component S_2.

Finally, calculating the inverse Fourier transform with respect to $\boldsymbol{v}$, we pass from (5.9) to the following expression:

$$\Phi(\mu, \boldsymbol{v}; t) = \left(\frac{\delta^{1/2}}{2\pi\tau \sinh \delta^{1/2}} \right)^{d/2} \exp\left(-\frac{\boldsymbol{v}^2 \delta^{1/2}}{2\tau \tanh \delta^{1/2}} \right) \tag{5.12}$$

for the Laplace transform

$$\Phi(\mu, \boldsymbol{v}; t) = \int_0^\infty \mathrm{e}^{-\mu s} \mathscr{P}_2(s, \boldsymbol{v}; t)\, \mathrm{d}s \tag{5.13}$$

of the probability density $\mathscr{P}_2(s, \boldsymbol{v}; t)$ with respect to the variable s.

Introduce an auxiliary dimensionless random variable

$$G_2 = S_2(\boldsymbol{y}, t)/2bt^2. \tag{5.14}$$

It follows from (5.10) that probability density $\tilde{\mathscr{P}}_2(g)$ is independent of time and has the Laplace transform

$$\tilde{\theta}_2(\delta) = \int_0^\infty \tilde{\mathscr{P}}_2(g) \mathrm{e}^{-\delta g} \mathrm{d}g = \frac{1}{\cosh \delta^{1/2}}. \tag{5.15}$$

Here, as in the previous section, we have taken $d = 2$. Using the inverse Laplace transform of (5.15) we get

$$\tilde{\mathscr{P}}_2(g) = \langle \delta(g - G_2) \rangle = \sum_{k=0}^\infty (-1)^k \frac{2k+1}{(\pi g)^{1/2} g} \exp\left(-\frac{(2k+1)^2}{4g} \right). \tag{5.16}$$

The probability density of the full, normed with respect to (5.14), action is equal to the convolution

$$\tilde{\mathscr{P}}(g;\tau) = \tilde{\mathscr{P}}_2(g) \otimes \tilde{\mathscr{P}}_1(g;\tau) \tag{5.17}$$

of the probability density (5.16), and the Gaussian probability density

$$\tilde{\mathscr{P}}_1(g;\tau) = \frac{1}{(2\pi\epsilon^2)^{1/2}} \exp\left(-\frac{g^2}{2\epsilon^2}\right), \tag{5.18}$$

obtained from (5.2) by passing to dimensionless variables τ and $g = s/2bt^2$. In (5.18), the dimensionless parameter

$$\epsilon = 2(ab/\tau^3)^{1/2}. \tag{5.19}$$

For sufficiently large times, when $\epsilon \ll 1$, the probability density $\mathscr{P}_1$ (5.18) plays the role of a delta-function in convolution (5.17), and we can use an approximate formula

$$\tilde{\mathscr{P}}(g) \approx \tilde{\mathscr{P}}_2(g). \tag{5.20}$$

5.2. *Asymptotics of the largest value of the least action*

The above discussion of statistical properties of the action of non-interacting particles will help us to find the largest value of least action $\bar{S}$ which, in turn, will determine statistical properties of Burgers' turbulence in the inviscid limit. Let us introduce, similar to (5.14), dimensionless value

$$\rho = \bar{S}/2bt, \tag{5.21}$$

For very large times, when $\epsilon \ll 1$ (see (5.2)) and additionally $\langle N(t)\rangle \gg 1$, it is sufficient to know the behavior of function (5.16) for small $g \ll 1$. For such g, the sum (5.16) is approximately equal to its first summand. As a result, we arrive at the asymptotic formula

$$\tilde{\mathscr{P}}(g) \sim \frac{1}{(\pi g)^{1/2} g} \exp\left(-\frac{1}{4g}\right), \qquad \epsilon \ll 1,\ g \ll 1. \tag{5.22}$$

Similarly, it is not difficult to show that in the space of arbitrary dimension d, the probability density of action is described by an asymptotic expression

$$\tilde{\mathscr{P}}(g) \sim \left(\frac{2^d}{\pi g}\right)^{1/2} \frac{d}{4g} \exp\left(-\frac{d^2}{16g}\right), \qquad \epsilon \ll 1,\ g \ll 1. \tag{5.23}$$

Consequently, the equation for ρ

$$\langle N(t)\rangle \int_0^\rho \tilde{\mathscr{P}}(g)\,\mathrm{d}g = 1 \tag{5.24}$$

assumes the form

$$\langle N(t)\rangle \sqrt{2^d}\, \operatorname{erfc}(-d/4\rho^{1/2}) = 1. \tag{5.25}$$

Utilizing the asymptotic formula (4.10), we can reduce (5.25) to the transcendental equation

$$\langle N(t)\rangle \frac{4}{d} \left(\frac{2^d\rho}{\pi}\right)^{1/2} \exp\left(-\frac{d^2}{16\rho}\right) = 1, \tag{5.26}$$

the asymptotic solution of which can be written in the form

$$\rho = d^2 \Big/ 16 \ln\left(\langle N(t)\rangle \left(\frac{2^{d+1}}{\pi}\right)^{1/2}\right). \tag{5.27}$$

5.3. *Average energy of Burgers' turbulence*

Now we can return to an analysis of the desired statistical characteristics of forced Burgers' turbulence. First of all, let us take a look at the behaviour of the average kinetic energy

$$\langle u(\boldsymbol{x}, t)\rangle = \tfrac{1}{2}\langle \boldsymbol{v}^2(\boldsymbol{x}, t)\rangle.$$

Multiply (5.12) by $\langle N(t)\rangle \boldsymbol{v}^2/2$ and then integrate it over all the values of $\boldsymbol{v}$. As a result, we obtain the following auxiliary function:

$$T(\delta, \tau) = \langle N(t)\rangle \tau d \left(\frac{1}{\cosh \delta^{1/2}}\right)^{d/2} \frac{\tanh \delta^{1/2}}{\delta^{1/2}}. \tag{5.28}$$

To calculate the average kinetic energy, it is necessary to find the inverse Laplace transform of that function with respect to δ, and then to integrate the obtained expression with respect to g, over the interval $(0, \rho)$. To implement these steps note that the behaviour of the desired original function for small values of g, which are of interest to us, is determined by the behaviour of its Laplace transform (5.28) for large values of δ. For that reason, we will pass in (5.28) to the corresponding asymptotic expression

$$T(\delta, \tau) \sim \langle N(t)\rangle d\tau \left(\frac{2^d}{\delta}\right)^{1/2} \mathrm{e}^{-(d/2)\delta^{1/2}} = -4\tau\langle N(t)\rangle \sqrt{2^d} \frac{\mathrm{d}}{\mathrm{d}\delta} \mathrm{e}^{-(d/2)\delta^{1/2}}, \qquad \delta \gg 1.$$

Finding the inverse Laplace transform of this function, integrating it over g in the interval $(0, \rho)$, we arrive at the following asymptotic formula for the kinetic energy of Burgers' turbulence:

$$\langle u(\boldsymbol{x}, t)\rangle \sim d\tau \langle N\rangle \left(\frac{2^d}{\pi}\right)^{1/2} \int_0^{\rho} \frac{\mathrm{d}g}{g^{1/2}} \exp\left(-\frac{d^2}{16g}\right).$$

Replacing the integral by its main asymptotics for $\rho \ll 1$, we have

$$\langle u(\boldsymbol{x}, t)\rangle \sim \tau\rho\langle N(t)\rangle \frac{16}{d} \left(\frac{2^d \rho}{\pi}\right)^{1/2} \exp\left(-\frac{d^2}{16\rho}\right).$$

Comparing this expression with equation (5.26) we finally obtain the following result:

(i) *Let* $\epsilon = 2(ab/\tau^3)^{1/2}$, $\tau = 2bt$, *and* $\langle N(t)\rangle$ *be the average number of streams of the auxiliary gas of non-interacting particles. Then, for* $\epsilon \ll 1$, $\langle N\rangle \gg 1$, *the average dimensionless kinetic energy in forced Burgers' turbulence has the following asymptotic behaviour:*

$$\langle u(\boldsymbol{x}, t)\rangle \approx 4\tau\rho, \tag{5.29}$$

where ρ *(5.27) is the largest possible value of the least action.*

The above conclusion and formula (5.27) give us an opportunity to formulate a necessary condition for existence of a stationary regime in forced Burgers' turbulence:

(ii) *A necessary condition for the existence of a stationary regime in forced Burgers' turbulence is the exponential growth*

$$\langle N(t)\rangle \sim C\mathrm{e}^{\gamma\tau} \tag{5.30}$$

of the average stream number in the auxiliary gas of non-interacting particles. The exponent γ *determines the limit average energy via the formula*

$$u_\infty = \lim_{t\to\infty} \langle u(\boldsymbol{x}, t)\rangle = d^2/4\gamma. \tag{5.31}$$

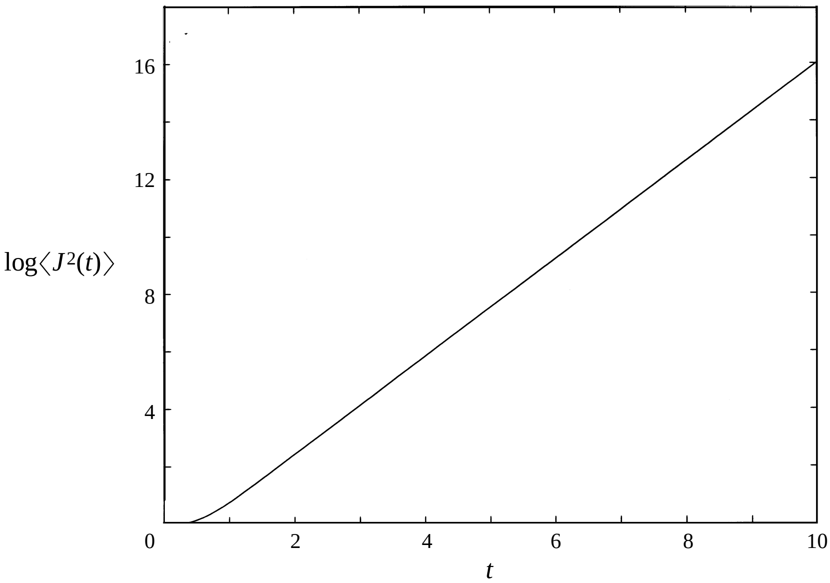

FIGURE 4. The logarithmic plot of the time evolution of the Jacobian's second moment. The exponential asymptotics is clearly visible.

In the multidimensional case, the verification of the exponential growth law (5.30) requires the knowledge of the joint $2d$-dimensional probability density $\mathscr{P}(\boldsymbol{j}, \boldsymbol{k}; t)$ for components of tensors $\boldsymbol{J}$ and $\boldsymbol{K}$. This is a formidable problem, both analytically and numerically. Nevertheless, linearity of the corresponding stochastic equations for $\boldsymbol{J}$ and $\boldsymbol{K}$ enables us to reach some conclusions about the behavior of the forced Burgers' turbulence for $t \to \infty$.

First of all, notice that in the two-dimensional case it is rather easy to derive the following exact equation for the second moment $\langle J^2 \rangle$ of the Jacobian:

$$\frac{\mathrm{d}^6 \langle J^2 \rangle}{\mathrm{d}\theta^6} - 14 \frac{\mathrm{d}^3 \langle J^2 \rangle}{\mathrm{d}\theta^3} - 2\theta \langle J^2 \rangle = 0, \tag{5.32}$$

where $\theta = c^{1/3} t$ is the dimensionless time, and c is the third coefficient in the power series expansion

$$a(y) = a - \frac{b}{2} y^2 + \frac{c}{8} y^4 - \dots$$

of function $a(y)$ from (3.1). A suitable solution of equation (5.32) has the form

$$\begin{aligned}\langle J^2 \rangle = &\left(1 - \sqrt{3/23}\right) \left[\exp(\beta_1 \theta) + 2 \exp(-\beta_1 \theta/2) \cos(\sqrt{3} \beta_1 \theta/2)\right] \\ &+ \left(1 + \sqrt{3/23}\right) \left[\exp(-\beta_2 \theta) + 2 \exp(\beta_2 \theta/2) \cos(\sqrt{3} \beta_2 \theta/2)\right],\end{aligned}$$

where $\beta_{1,2} = \sqrt{\sqrt{69} \pm 7}$, and it grows monotonically with θ. As $\theta \to \infty$, we have the exponential asymptotics

$$\langle J^2 \rangle \sim (1 - \sqrt{3/23}) \mathrm{e}^{\beta_1 \theta}$$

(see figure 4).

In addition, it is also clear that

$$\langle N(t) \rangle = \langle |J| \rangle < \langle J^2 \rangle^{1/2} \sim \exp(\beta_1 \theta/2). \tag{5.33}$$

It means that the *average energy of forced Burgers' turbulence is bounded from below and satisfies the following asymptotic inequality:*

$$\langle u(\boldsymbol{x}, t)\rangle \geqslant d^2 b/\beta_1 c^{1/3}. \tag{5.34}$$

Remark 5.1. If the self-similarity property (4.19) is taken as a working hypothesis (it has been established in the previous section for unforced Burgers' turbulence), then the exponential law (5.30) follows with

$$\gamma = \beta_1 c^{1/3}/4b,$$

and the kinetic energy converges to the stationary value

$$u_\infty = d^2 b/\beta_1 c^{1/3}$$

which coincides with the right-hand side of bound (5.34). The one-dimensional case, where the crucial exponential law (5.30) can be derived by more precise methods, will be discussed in the next section.

6. Stream-number statistics for a one-dimensional gas of non-interacting particles

In this section we discuss statistical properties of the Jacobian (3.11) and find an asymptotic rate of growth of the average number of streams $\langle N(t)\rangle$ (3.29). We will restrict our attention to the one-dimensional case. Then, equations for the Jacobian (3.4) have the particularly simple form

$$\frac{\mathrm{d}J}{\mathrm{d}t} = K, \qquad \frac{\mathrm{d}K}{\mathrm{d}t} = g(X, t)J. \tag{6.1}$$

In the delta-correlated approximation used in this paper, the random field $g(x, t)$ can be replaced by a statistically equivalent Gaussian process $g(t)$ with zero mean and correlation function

$$\langle g(t)g(t+\theta)\rangle = 2c\delta(\theta). \tag{6.2}$$

We need to solve equations (6.1) with initial conditions

$$J(t=0) = 1, \qquad K(t=0) = 0. \tag{6.3}$$

Let us introduce an ordered sequence

$$0 < t_1 < t_2 < \ldots < t_m < \ldots \tag{6.4}$$

of times $\{t_m\}$ which are roots of the equation

$$J(t) = 0. \tag{6.5}$$

Take one of these times t_m as the initial time. Then, the solution of equation (6.1) sought for $t > t_m$ can be written in the form

$$J(t) = \tilde{K}(t_m)\tilde{J}(t|t_m), \qquad K(t) = \tilde{K}(t_m)\tilde{K}(t|t_m), \tag{6.6}$$

where $\tilde{J}(t|t_m)$ and $\tilde{K}(t|t_m)$ are solutions of equation (6.1) with the initial conditions

$$\tilde{J}(t=t_m|t_m) = 0, \qquad \tilde{K}(t=t_m|t_m) = 1. \tag{6.7}$$

Expressing, in turn, $\tilde{K}(t_m)$ by $\tilde{K}(t_{m-1})$ and so on, we arrive at the equality

$$\tilde{K}(t_m) = \prod_{p=1}^{m} K_p, \tag{6.8}$$

where

$$K_1 = K(t_1), \qquad K_p = \tilde{K}(t_p|t_{p-1}), \quad p > 1.$$

Additionally, observe that – according to (6.6) – the product of random variables (6.8) defines the value $J(t)$ of the solution of the initial value problem (6.1)–(6.3) at time $t > t_m$:

$$J(t) = \tilde{J}(t|t_m) \prod_{p=1}^{m} K_p. \tag{6.9}$$

We emphasize that, for a given value of m, all the factors in the products (6.8)–(6.9) are statistically mutually independent, since they are functionals of the white noise $g(t)$ on the non-overlapping time intervals (t_{p-1}, t_p). It is not difficult to show that an even more general statement is true: elements of the sequence of random quantities $\{K_p, \tau_p\}$, where

$$\tau_p = t_p - t_{p-1},$$

with different indices p and p' are statistically independent, and the joint probability density with identical indices

$$w(\kappa, \tau) = \langle \delta(K_p - \kappa)\delta(\tau_p - \tau)\rangle, \qquad p > 1,$$

does not depend on the index p.

Recall that, in the final count, we are interested in the average stream-number $\langle N(t)\rangle$ (3.29)

$$\langle N(t)\rangle = \langle |J(t)|\rangle. \tag{6.10}$$

For sufficiently large times, when $\langle N(t)\rangle \gg 1$, using the law of large numbers one can assume that

$$m = t/\langle \tau_1 \rangle, \tag{6.11}$$

where $\langle \tau_1 \rangle$ is the mean length of the time interval between adjacent zeros of the process $J(t)$. In this fashion, taking into account (6.9), we obtain the following conclusion

Conclusion. *In forced one-dimensional Burgers' turbulence, the average stream number*

$$\langle N(t)\rangle \sim C e^{\nu t}, \qquad t \gg \langle \tau_1 \rangle,$$

where the exponent

$$\nu = \frac{1}{\langle \tau_1 \rangle} \ln\left(\langle K \rangle\right), \tag{6.12}$$

and $\langle K \rangle$ *is the statistical average of any of the random factors in the product (6.8) for* $p > 1$.

Hence, the calculation of the exponent ν reduces to finding the averages $\langle \tau_1 \rangle$ and $\langle K \rangle$. These averages can be computed numerically. For that purpose we introduce a new dimensionless time

$$\theta = c^{1/3} t,$$

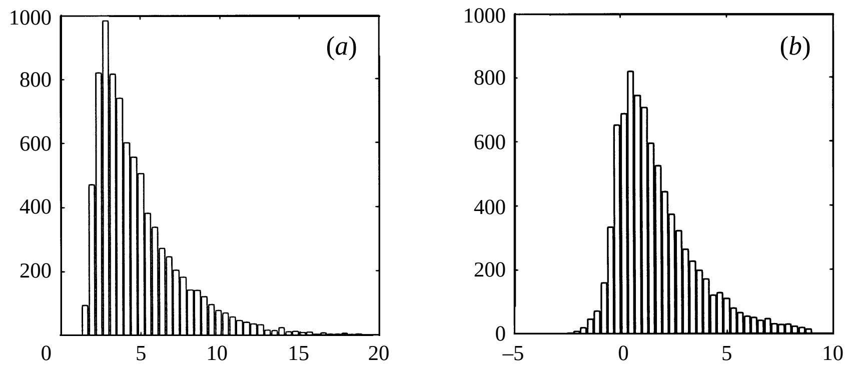

FIGURE 5. The histograms of data $\{\theta_1^m\}$ (*a*) and $\{\log K^m\}$ (*b*) for $M \approx 8000$. The logarithmic scale was needed in case (*b*) because of the huge variance of the data

and transform (6.1) into dimensionless equations

$$\frac{\mathrm{d}R}{\mathrm{d}\theta} = K, \qquad \frac{\mathrm{d}K}{\mathrm{d}\theta} = \alpha(\theta)R, \tag{6.13}$$

where $\alpha(\theta)$ is a Gaussian, delta-correlated process with correlation function

$$\langle \alpha(\theta)\alpha(\theta + \eta)\rangle = 2\delta(\eta).$$

The suggested scheme of numerical calculations of $\langle \tau_1 \rangle$ and $\langle K \rangle$ requires repeated numerical solutions of equations (6.13) with initial conditions

$$R(0) = 0, \quad K(0) = 1,$$

for a large number $M \gg 1$ of statistically independent realizations of $\alpha(\theta)$. Stopping the calculations at the first moment $\theta = \theta_1 > 0$ when $R_1(\theta_1) = 0$, we obtain two data arrays $\{\theta_1^m\}$ and $\{K^m\}$, $m = 1, 2, \ldots, M$, $K^m = K^m(\theta_1^m)$, the means of which give us approximate values of statistical averages of θ_1 and K. Notice that $\langle \theta_1 \rangle$ is related to the above-mentioned average $\langle \tau \rangle$ via an obvious equality

$$\langle \tau_1 \rangle = \langle \theta_1 \rangle c^{-1/3}.$$

The histograms on figure 5 illustrate the results of $M \approx 8000$ such numerical calculations. In particular, they provide the following estimates: $\langle \theta_1 \rangle \approx 4.83$, $\langle K \rangle \approx 81.26$, and as a result

$$v \approx 0.91 c^{1/3}.$$

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Appendix A. The Furutsu–Novikov–Donsker formula applied to forced Burgers' turbulence

The Furutsu–Novikov–Donsker formula

$$\langle f_i(\boldsymbol{x})R[f]\rangle = \int \langle f_i(\boldsymbol{x})f_k(\boldsymbol{x}')\rangle \left\langle \frac{\delta R[\boldsymbol{f}]}{\delta \boldsymbol{f}(\boldsymbol{x}')\mathrm{d}\boldsymbol{x}} \right\rangle \mathrm{d}\boldsymbol{x}',$$

where $\delta R[\boldsymbol{f}]/\delta \boldsymbol{f}$ is the variational derivative of the functional R, introduced by Furutsu (1963) in the context of the statistical theory of electromagnetic waves in a fluctuating medium, and by Novikov (1964) in a study of randomly forced turbulence, is a powerful tool in the analysis of random processes and fields. Donsker (1964) obtained it independently while studying the mathematical theory of path integrals. The formula explicitly calculates the correlation of an arbitrary zero-mean Gaussian field $\boldsymbol{f}(\boldsymbol{x}) = (f_i(\boldsymbol{x}))_i$ and its analytic functional $R[\boldsymbol{f}]$, and is obtained by a straightforward formal comparison of the functional power series expansions of the left-hand side and the right-hand side.

We will illustrate its usefulness by applying it to evaluate the correlation $\langle f(x,t)\upsilon(x,t)\rangle$, where $f(x,t)$ is a Gaussian random field with zero mean and correlation function (1.7), and $\upsilon(x,t)$ is the solution of one-dimensional Burgers' equation (1.5). In this case, the Furutsu–Novikov–Donsker formula yields the following exact equality:

$$\langle f(x,t)\upsilon(x+z,t)\rangle = \int \mathrm{d}y \int_0^t \mathrm{d}\tau \langle f(x,t)f(y,\tau)\rangle \left\langle \frac{\delta\upsilon(x+z,t)}{\delta f(y,\tau)} \right\rangle. \tag{A 1}$$

Applying the variational derivative it to Burgers' equation (1.5) gives

$$\frac{\partial}{\partial t}\left(\frac{\delta\upsilon(x,t)}{\delta f(y,\tau)}\right) + \frac{\partial}{\partial x}\left(\upsilon(x,t)\frac{\delta\upsilon(x,t)}{\delta f(y,\tau)}\right) = \mu\frac{\partial^2}{\partial x^2}\frac{\delta\upsilon(x,t)}{\delta f(y,\tau)} + \delta(x-y)\delta(t-\tau).$$

Now, taking into account the causality principle, one can replace the above linear equation for the variational derivative sought by the following Cauchy problem for the homogeneous equation:

$$\left.\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\delta\upsilon(x,t)}{\delta f(y,\tau)}\right) + \frac{\partial}{\partial x}\left(\upsilon(x,t)\frac{\delta\upsilon(x,t)}{\delta f(y,\tau)}\right) = \mu\frac{\partial^2}{\partial x^2}\frac{\delta\upsilon(x,t)}{\delta f(y,\tau)}, \\ \frac{\delta\upsilon(x,t=\tau)}{\delta f(y,\tau)} = \delta(x-y). \end{aligned}\right\} \tag{A 2}$$

Substituting into (A1) the correlation function (1.7), we obtain

$$\langle f(x,t)\upsilon(x+z,t)\rangle = \int \mathrm{d}y\, \Gamma_f(y-x) \int_0^t \delta(t-\tau) \left\langle \frac{\delta\upsilon(x+z,t)}{\delta f(y,\tau)} \right\rangle \mathrm{d}\tau,$$

or, using the probing property of the Dirac delta,

$$\langle f(x,t)\upsilon(x+z,t)\rangle = \frac{1}{2}\int \mathrm{d}y\Gamma_f(y-x) \left\langle \frac{\delta\upsilon(x+z,t)}{\delta f(y,t)} \right\rangle.$$

So, finally, in view of equality (A2),

$$\langle f(x,t)\upsilon(x+z,t)\rangle = \frac{1}{2}\Gamma_f(z).$$

Appendix B. Mechanism of energy dissipation in inviscid one-dimensional Burgers' turbulence

In this Appendix we discuss, in the relatively simple one-dimensional case, the mechanism of energy dissipation in inviscid Burgers' turbulence and the corresponding problems of steady-state regimes maintained in the presence of external forces.

First, let us consider the homogeneous Burgers' equation

$$\frac{\partial \upsilon}{\partial t} + \upsilon \frac{\partial \upsilon}{\partial x} = \mu \frac{\partial^2 \upsilon}{\partial x^2},$$

$$\upsilon(x, t = 0) = \upsilon_0(x),$$

where $\upsilon_0(x)$ is a stationary and homogeneous random field. Then, obviously, the solution $\upsilon(x, t)$ of this equation is also a statistically homogeneous function of x. This means that the average energy

$$\langle u(x, t) \rangle = \tfrac{1}{2} \left\langle \upsilon^2(x, t) \right\rangle$$

obeys to the equation

$$\frac{\mathrm{d} \langle u \rangle}{\mathrm{d}t} = -\bar{\epsilon}, \tag{B 1}$$

where the energy dissipation rate is defined by the formula

$$\bar{\epsilon} = \mu \left\langle g^2(x, t) \right\rangle, \tag{B 2}$$

where

$$g(x, t) = \frac{\partial \upsilon(x, t)}{\partial x}$$

is Burgers' turbulence velocity gradient. It is clear from (B2) that, in the inviscid limit $\mu \to 0_+$, dissipation occurs only in the infinitesimal vicinities of Burgers' velocity shock fronts, where the velocity gradient has big jumps of size $\sim 1/\mu$. These large peaks balance the influence of the vanishing coefficient μ at the right-hand side of (B2).

To recover the detailed mechanism of energy dissipation in the inviscid limit let us recall (see e.g. Gurbatov *et al.* 1991) the universal shape of Burgers' equation's solution in a small vicinity of the shock front of size a, moving with velocity V, and situated at the point $x^* = x - Vt + C$:

$$\upsilon_s(x - x^*, a) = V - \frac{a}{2} \tanh\left(\frac{a(x - x^*)}{4\mu}\right).$$

The corresponding velocity field gradient has, in the vicinity of this shock, the form

$$g_s(x - x^*, a) = -\frac{a^2}{8\mu} \frac{1}{\cosh^2\left(a(x - x^*)/4\mu\right)}. \tag{B 3}$$

It is physically natural to assume that, for sufficiently small viscosity μ, the gradient is of the same shape in the case of a forced Burgers' velocity field. So, neglecting the contribution to the dissipation rate of the gradient field realizations in between shocks, we can write these realizations in the form of a series of non-overlapping peaks:

$$g(x, t) = \sum_k g_s(x - x_k, a_k), \tag{B 4}$$

where x_k and a_k are coordinates and amplitudes of successive shocks. Substituting (B4) into (B2), and taking into account (B3), we get

$$\bar{\epsilon} = \left\langle \frac{\vartheta a^4}{64\mu} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\cosh^4\left(ax/4\mu\right)} \right\rangle,$$

where $\vartheta(a,t)$ denotes the average spatial frequency of shocks with amplitude a at the time t, and angle brackets denote statistical averaging over random shock amplitudes a_k. Evaluating the integral we get

$$\bar{\epsilon} = \frac{\left\langle \vartheta a^3 \right\rangle}{12}.$$

For the forced one-dimensional Burgers' equation (1.5) and delta-correlated Gaussian forces (1.7), in view of (1.9) (see also Appendix A), the average energy obeys an equation similar to (B):

$$\frac{\mathrm{d}\langle u\rangle}{\mathrm{d}t} = -\bar{\epsilon} + \tfrac{1}{2}\Gamma_f,$$

where $\Gamma_f = \Gamma_f(z=0)$. At the initial stage, when shocks are virtually absent ($\vartheta \approx 0$), we get

$$\frac{\mathrm{d}\langle u\rangle}{\mathrm{d}t} = \tfrac{1}{2}\Gamma_f,$$

and the energy of turbulence is increasing linearly:

$$< u > \approx \tfrac{1}{2}t\Gamma_f.$$

Then the growth rate of $\bar{\epsilon}$ is reduced due the appearance of shock fronts in Burgers' velocity field realizations. Eventually, for the steady-state regime of forced Burgers' turbulence, the frequency of shocks, their amplitudes and the statistical properties of external forces are tied by the equality

$$\left\langle \vartheta a^3 \right\rangle = 6\Gamma_f.$$

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